Arithmetic -modules and Representations

King Fai Lai

ARITHMETIC $\mathcal{D}$-MODULES AND REPRESENTATIONS

KING FAI LAI

Abstract. We propose in this paper an approach to Breuil's conjecture on a Langlands correspondence between p-adic Galois representations and representations of p-adic Lie groups in p-adic topological vector spaces. We suggest that Berthelot's theory of arithmetic D-modules should give a p-adic analogue of Kashiwara's theory of D-modules for real Lie groups i.e. it should give a realization of the p-adic representations of a p-adic Lie group as spaces of overconvergent solutions of arithmetic D-modules which will come equipped with an action of the Galois group. We shall discuss the case of Siegel modular varieties as a possible testing ground for the proposal.

1. Introduction

Breuil conjectured that there is a Langlands correspondence between p-adic Galois representations and representations of p-adic Lie groups in p-adic topological vector spaces. Breuil-Schneider ([BrS 03]) pointed out that at the moment it is difficult to construct p-adic representations of a p-adic Lie group.

In this note I propose an approach to this problem, namely, use Berthelot's theory of arithmetic D-modules to give a p-adic analogue of Kashiwara's theory of D-modules for real Lie groups i.e. we want to realize p-adic representations of a p-adic Lie group as solution spaces of arithmetic D-modules which will come equipped with an action of the Galois group. This is described in the third section. In the last section we consider the case of Siegel modular varieties as a possible testing ground for the proposal.

As this note is based on talks to audiences with diverse backgrounds, I mention facts which may be known to one group but not to another. In any case this is a dream so no apologies to experts.

At this point it remains for me to thank Christoph Breuil, Mat Emerton, Paul Gerardin, Michel Gros, Dick Gross, C-G. Schmidt, Jing Yu for conversations on this note, Pierre Berthelot for interest in and support for this fantasy, Raja Varadarajan for his help with the hypergeometric equation and Hotta Ryoshi for sending his Indian notes to an unknown foot soldier.

2000 *Mathematics Subject Classification.* Primary:11F80, 12H25, 11F33, 11F46, 14F10,14F30, 13D07, 13D25, 32C38 .

Key words and phrases. arithmetic D modules, representations of p-adic group, p-adic modular forms .

Acknowledgement: These are notes based on talks I gave at the the Mathematics Institute of NCTS at Tsing Hua, Taiwan (on 9 May 2007), the Institute of Mathematics at the University Rennes (IRMAR) (on 29 June 2007) and the Mathematics Institute of Karlsruhe University (on 4 July 2007). Their generous hospitalities are gratefully acknowledged.

2. In the beginning

2.1. **Over $\mathbb{R}$.** Let me begin with the discrete series of $G = SL_2(\mathbb{R})$ (Bargmann [Bar 47]). The Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ of $SL_2(\mathbb{R})$ consists of 2×2 matrices of trace 0 and is generated by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The center of the universal enveloping algebra $U(\mathfrak{g})$ is generated by the Casmir operator :

$$\Omega = (H+1)^2 + 4YX$$

We identify elements of the universal enveloping algebra $U(\mathfrak{g})$ with differential operators on G. If we use polar coordinates via the Cartan decomposition of G:

$$\phi : (\theta_1, t, \theta_2) \mapsto u_{\theta_1} a_t u_{\theta_2}$$

where $u_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ then $d\phi$ is bijective and a differential operator D on G goes to the differential operator $\tilde{D}$. For Ω this gives

$$\tilde{\Omega} = \frac{1}{\sinh^2 2t}\left(\frac{\partial^2}{\partial\theta_1^2} + \frac{\partial^2}{\partial\theta_2^2}\right) - 2\frac{\cosh 2t}{\sinh^2 2t}\frac{\partial^2}{\partial\theta_1\partial\theta_2} + \frac{\partial^2}{\partial t^2} + 2\frac{\cosh 2t}{\sinh 2t}\frac{\partial}{\partial t} + 1$$

Let $k \geq 1$ and π_k be the discrete series with lowest weight $k+1$ and infinitesimal character χ_k i.e. $\chi_k(\Omega) = k^2$. We can realize this representation in $L^2(G)$. Let us take a square integrable f satisfying

$$f(u_{\theta_1} x u_{\theta_2}) = e^{i(m\theta_1 + n\theta_2)} f(x).$$

Since

$$dg = \frac{1}{2}\sinh 2t d\theta_1 dt d\theta_2$$

we put $\tilde{f}(t) = (\sinh 2t)^{\frac{1}{2}} f(a_t)$. Then in polar coordinates the Casmir operator $\tilde{\Omega}$ goes to the operator $\Delta_{mn} = d^2/dt^2 - q_{mn}$ with

$$q_{mn} = \frac{m^2 + n^2 - 2mn\cosh 2t - 1}{\sinh^2 2t}.$$

Then function $\tilde{f}$ then satisfies the differential equation

$$\Delta_{mn}\tilde{f} = k^2 \tilde{f}.$$

If we go to a new variable $u = \cosh 2t$ then the differential equation becomes

$$\frac{d^2 g}{du^2} + \frac{P(u)}{u^2 - 1}\frac{dg}{du} + \frac{Q(u)}{(u^2-1)^2} g = 0$$

where P, Q are polynomials in u and deg $P \leq 1$, deg $Q \leq 2$. This is a hypergeometric equation with singularities at $u = 1, -1, \infty$ - such equations are indeed very familiar in p-adic analysis which has been studied by Dwork and his school ([Dwo 73], [Put 86], [Ked 05]). Notice there is one equation for each pair (m, n) i.e. for each K type - this will be important when we put coherence conditions on admissibility to insure some holonomy. In addition for something closer to modular forms I would like to draw attention to the Halphen Fricke differential operators and its application to p-adic modular forms in [Kat 76] (in particular §2.4.5). For more differential operators on automorphic forms see [Shi 81] and its applications to p-adic Siegel modular forms see [Pan 04].

We have the usual action of G on the upper half plane $\mathfrak{h}$ by :

$$g \cdot z = \frac{az+b}{cz+d}$$

with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $z \in \mathfrak{h}$. The discrete series representation π_k can also be realized in the space of holomorphic functions $f : \mathfrak{h} \to \mathbb{C}$ on the upper plane $\mathfrak{h}$ which are square integrable with respect to the measure $y^{k-2}dxdy$. The action of $SL_2(\mathbb{R})$ is given by

$$\left(\pi_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} f\right)(z) = (-cz+a)^{-k} f(\frac{dz-b}{-cz+a})$$

for $z \in \mathfrak{h}$. If we put

$$e_n(z) = (z-i)^{n-k-1}(z+i)^{1-n} \ ,$$

then the space of this representation is the completed Hilbert sum of e_n for $n \geq k+1$. This is often referred to as the holomorphic discrete series.

2.2. **The p adic case.** Next we consider the p-adic representations of Berger-Breuil. Let E be an extension of $\mathbb{Q}_p$. For a positive real number r, say a function $f : \mathbb{Z}_p \to E$ is in $\mathcal{C}^r$ if $n^r|a_n(f)| \to 0$ in $\mathbb{R}$ where $a_n(f) = \sum_{i=0}^{n}(-1)^i \begin{pmatrix} n \\ i \end{pmatrix} f(n-i)$. Note that $f(z) = \sum_{n=0}^{\infty} a_n(f) \begin{pmatrix} z \\ n \end{pmatrix}$ and that f is in $\mathcal{C}^n$ if and only if f has *continuous* p-adic derivative of order up to n.

Normalize the valuation by $val(p) = 1$. Write the ring of integers of E as $\mathcal{O}_E$. Take $\alpha \neq \beta$ in $\mathcal{O}_E$. Let $B(\alpha)$ be the space of functions $f : \mathbb{Q}_p \to E$ such that the restrictions $f|\mathbb{Z}_p$ is in $\mathcal{C}^{val(\alpha)}$ and $(\alpha p \beta^{-1})^{val(z)} z^{k-2} f(1/z)|\mathbb{Z}_p$ can be extended to a function in $\mathcal{C}^{val(\alpha)}$. Then $GL_2(\mathbb{Q}_p)$ acts in the following manner:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = \alpha^{val(ad-bc)}(\alpha p \beta^{-1})^{val(-cz+a)}(-cz+a)^{k-2} f(\frac{dz-b}{-cz+a}).$$

Let $L(\alpha)$ be the closed subspace of $B(\alpha)$ spanned by z^j, $(\alpha p \beta^{-1})^{val(z-a)}(z-a)^{k-2-j}$ for $a \in \mathbb{Q}_p$, $j \in \mathbb{Z}$, $0 \leq j < val(\alpha)$. Put $\Pi = B(\alpha)/L(\alpha)$. Then it is a deep theorem of Berger-Breuil that $\Pi \neq 0$ and that Π is an admissible representation od $GL_2(\mathbb{Q}_p)$.

Looking at the action defined by Berger-Breuil and the action in the holomorphic discrete series of $SL_2(\mathbb{R})$ one could not help asking if one could write down the differential equation in this case or if it is possible to construct their model using arithmetic theory of $\mathcal{D}$-modules. In fact there is a differential equation lurking behind the model of Berger-Breuil; in [BBr 06] §5.2 using the Fontaine functor ([Fon 90] A 3.4) the dual of the representation Π above is shown to be isomorphic to the space of bounded sequences in the inverse limit of (ϕ, Γ)-modules constructed from the starting data α, β and the inverse of Berger functor would take a (ϕ, Γ)-module to the solution space of a differential equation ([Beg 04] Theorem A and

C). The study of the differential equation on the boundary of the unit dics reminds me of the asymptotics at infinity of invariant eigendistributions of Harish-Chandra.

2.3. **Philosophy.** Before I leave SL_2 I turn to a bit of general background. The theory of Langlands correspondence says there is a bijection between the representations π of a reductive group G over a field F into the automorphism group of a vector space over a field E and the representations ρ of a Galois group of the field F into the Langlands dual of G which "almost" matches the Frobenius eigenvalues of ρ with the spherical Hecke eigenvalues of π. This has been established in the cases of $G = GL_n$, $E = \mathbb{C}$ and F is a a local field field, a one variable function field over a finite field or over the complex numbers by Langlands-Deligne-Kutzko-Harris-Taylor-Henniart, Laumon-Rapoport-Stuhler, Drinfeld-Lafforgue, Frenkel-Gaitsgory-Vilonen respectively ([Car 00], [LRS 93], [Lau 02],[Fre 04]) and almost for $G = GL_2$, $E = \mathbb{C}$, F a totally real filed by Langlands-Deligne-Carayol-Saito. (Readers will object to the lack of precision of the description here - in case F is a number field one should say π is an automorphic representation and ρ is a compatible system of λ adic representations; or one should mention the Weil-Deligne group, the automorphic Langlands group ([Art 02]), the motivic Galois group; one should be more precise about the representations occurring and mention automorphic forms, the L-indistinguishability of Labesse-Langlands, endoscopy of Kottwitz-Shelstad; one should point out the Deligne-Laumon geometric interpretation with Galois group as fundamental group and using Grothendieck's function-sheaf dictionary on the bijection between $GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})$ isomorphism classes of rank n vector bundles. But this would be besides the point here.)

Reading the proofs of the known correspondences one could not help having a feeling of staring at a magical inventory in a gigantic Amazon warehouse. The purpose of this note is to propose a "natural" setting for the Langlands correspondence, namely, the Galois group and the reductive group occurs as mutual centralizers in the monodromy group of a system of differential equations. This might not sound so far fetched from the points of view of Grothendieck's theory of monodromy and Harish-Chandra's constructions of representations using differential equations.

2.4. **Harish-Chandra.** Let G be a semisimple Lie group with finite center and Γ be a discrete subgroup of G such that $\Gamma\backslash G$ has finite volume. Then G acts on the Hilbert space of square integrable functions $L^2(\Gamma\backslash G)$ by translation $f(x) \mapsto f(g^{-1}x)$ giving the so called the regular representation of G. A basic theme in the theory of automorphic forms is to study the representations of G that occur in the spectral decomposition of this regular representation. For this we need to know the representations of G. This is done by Harish-Chandra using the theory of differential equations. Differential equations comes in right from the beginning in the paper of Bargmann which was extended by Harish-Chandra to all bounded symmetric domains, this line leads eventually to Schmidt's solution of Langlands conjecture constructing representations on L^2-cohomology of line bundles.

One more remark to make is this: following the tradition of Local-Global principle in Classfield theory, we replace $\Gamma\backslash G$ by $G(\mathbb{Q})\mathcal{G}(\mathbb{A})$ for a reductive group G over

$\mathbb{Q}$ and study representations of the form $\otimes\pi_p$ with π_p a representation of $G(\mathbb{Q}_p)$ on complex vector spaces (here: for $p = \infty$ we take $\mathbb{Q}_p$ to be $\mathbb{R}$). Again Harish-Chandra has constructed these representations for finite p.

2.5. **D-modules.** Nowadays beginning with Beilinson-Bernstein we discuss the representation theory of real Lie groups in the frame work of D-modules.

For a complex analytic (or algebraic) manifold X let $\mathcal{D}_X$ be the sheaf of analytic differential operators on X ([Bjo 93] §1.2.2; or [Bor 87] II 2.12, VI 1.1, 1.2). On an open subset U of X, to an differential operator Δ in $\mathcal{D}_X(U)$ we associate the $\mathcal{D}_X(U)$-module $M + \mathcal{D}_X(U)/\Delta\mathcal{D}_X(U)$. Then we can think of $Hom_{\mathcal{D}_X(U)}(M, \emptyset_X(U))$ as the space of solutions of the differential equations $\Delta = 0$ on U. This will be our object of interest.

Partly to supplement the claim that "representations of a semisimple Lie group are realized as solution spaces of systems of differential equations" I recall some results of Kashiwara - Schmid company.

Let G be a semisimple Lie group with finite center with Lie algebra $\mathfrak{g}$. Let K be the maximal compact subgroup of G. Let $Mod(\mathfrak{g}_{\mathbb{C}}, K)$ denote the category of algebraic $(\mathfrak{g}_{\mathbb{C}}, K)$-modules; this category contains the Harish-Chandra modules which correspond to admissible representations of G. Write $D^b(Mod(\mathfrak{g}_{\mathbb{C}}, K))$ for the bounded derived category of $Mod(\mathfrak{g}_{\mathbb{C}}, K)$. Let $Mod_G(\mathcal{D}_{G/K})$ denote the category of quasi coherent G equivariant $\mathcal{D}_{G/K}$-modules. Write $D^b_G(\mathcal{D}_{G/K})$ for the bounded derived category of $Mod_G(\mathcal{D}_{G/K})$. The Kashiwara showed that the categories $D^b(Mod(\mathfrak{g}_{\mathbb{C}}, K))$ and $D^b_G(\mathcal{D}_{G/K})$ are equivalent.

Again let G be a semisimple Lie group with finite center with Lie algebra $\mathfrak{g}$. Write $\mathcal{U}(\mathfrak{g})$ for the universal enveloping algebra of $\mathfrak{g}$. Fix a Borel subgroup B of G. Put $\mathcal{U}_{G/B}(\mathfrak{g}) = \mathcal{O}_{\mathcal{G}/B} \otimes \mathcal{U}(\mathfrak{g})$ with multiplication such that $\mathfrak{g}$ acts on $\mathcal{U}(\mathfrak{g})$ by left multiplication and on $\mathcal{O}_{\mathcal{G}/B}$ by differentiation. Let $\tilde{\mathfrak{g}}$ be the kernel of the morphism from $\mathcal{U}_{G/B}(\mathfrak{g})$ to the sheaf of vector fields on G/B. Fix a maximal torus T in B. Write the Lie algebra of T as $\mathfrak{t}$. Let λ be a $\mathbb{C}$-algebra homomorphism from the center $\mathfrak{z}$ of $\mathcal{U}(\mathfrak{g})$ to $\mathbb{C}$ which we also identify with an element of the dual $\mathfrak{t}^*$ and extend to an element of $\mathfrak{b}^*$. We get a one dimensional representation $\mathbb{C}_\lambda = \mathbb{C} \cdot 1_\lambda$ of $\mathfrak{b}^*$ by $A \cdot 1_\lambda = \lambda(A)\ 1_\lambda$ for $A \in \mathfrak{b}$. Set

$$\underline{A}_{G/B}(\lambda) = \mathcal{U}_{G/B}(\mathfrak{g}) / \sum_{A \in \tilde{\mathfrak{g}}} \mathcal{U}_{G/B}(\mathfrak{g})(A - \lambda(A)).$$

Let ρ be half sum of the positive roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ with respect to $\mathfrak{b}$. The ring $\mathcal{D}_{G/B,\lambda} = \underline{A}_{G/B}(\lambda + \rho)$ is Kashiwara's twisted ring of differential operators on G/B and he proved in this case a Riemann-Hilbert correspondence which says that $\mathcal{M} \mapsto R\mathcal{H}om_{\mathcal{D}_{G/B,\lambda}}(\mathcal{O}_{G/B} \otimes \mathbb{C}_\lambda, \mathcal{M})$ yields an equivalence from the bounded derived category of regular holonomic $\mathcal{D}_{G/B,\lambda}$-modules to the bounded derived category of constructible sheafs of complex vector spaces on G/B twisted by $-\lambda$.

Before I forget I would like to mention the work of Hotta ([Hot 87]) on character sheaves (and the related work in the p adic case of Gros [Gro 01], Aubert-Cuningham [AuC 04]) and on Morihihiko Saito (and the related comments of Berthelot in the case of arithmetic modules).

3. Homogeneous Formal Schemes

3.1. **Equivariant Arithmetic $\mathcal{D}$-modules.** Fix a prime p. Let $\mathcal{V}$ be a complete discrete valuation ring of characteristic zero with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{F}$ of characteristic p. Write F for the field of fraction of $\mathcal{V}$.

Let $\mathfrak{X}$ be a locally noetherian proper smooth formal $\mathcal{V}$ scheme. we always assume that $\mathfrak{m}\mathcal{O}_{\mathfrak{X}}$ is an ideal of definition.

Let $\mathfrak{G}$ be an affine formal group $\mathcal{V}$ scheme acting homogeneously on $\mathfrak{X}$ over $\mathcal{V}$. Let us write the action map as $m : \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X} : g, x \mapsto gx$ and the projection as $p : \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X}$. Define maps $d_j : \mathfrak{G} \times \mathfrak{G} \times \mathfrak{X} \to \mathfrak{G} \times \mathfrak{X}$ by $d_1(g_1, g_2, x) = (g_1, g_2x)$, $d_2(g_1, g_2, x) = (g_1g_2, x)$, $d_3(g_1, g_2, x) = (g_2, x)$. Then our assumption says $md_1 = md_2$, $pd_2 = pd_3$, $md_3 = pd_1$. To begin with this action would allow us to consider the elements of the universal enveloping algebra of $\mathfrak{G}$ as differential operators on $\mathfrak{X}$.

Let $\mathcal{D}^{\dagger}_{\mathfrak{X}}$ the sheaf of differential operators of infinite order of finite level on $\mathfrak{X}$ introduced by Berthelot ([Ber 961] (2.4.1)). By a $\mathfrak{G}$-equivariant $\mathcal{D}^{\dagger}_{\mathfrak{X}}$ module we mean a pair consisting a $\mathcal{D}^{\dagger}_{\mathfrak{X}}$ module $\mathcal{M}$ together with a $\mathcal{D}^{\dagger}_{\mathfrak{G}\times\mathfrak{X}}$ linear isomorphism $b : m^*\mathcal{M} \to p^*\mathcal{M}$ such that we have the following commutative diagram

$$\begin{array}{ccc} d_2^*m^*\mathcal{M} & \xrightarrow{d_2^*b} & d_2^*p^*\mathcal{M} \\ \big\downarrow & & \big\downarrow \\ d_1^*m^*\mathcal{M} & \xrightarrow{a} & d_3^*p^*\mathcal{M} \end{array}$$

where a is the composite $d_1^*m^*\mathcal{M} \xrightarrow{d_1^*b} d_1^*p^*\mathcal{M} \to d_3^*m^*\mathcal{M} \xrightarrow{d_3^*b} d_3^*p^*\mathcal{M}$.

Let $Mod_{\mathfrak{G}}(\mathcal{D}^{\dagger}_{\mathfrak{X}})$ denote the category of *admissible* $\mathfrak{G}$-equivariant $\mathcal{D}^{\dagger}_{\mathfrak{X}}$ modules with $\mathfrak{G}$ equivariant morphisms between them. We have left open the requirements that should be put into the word *admissible*. For example we would want $Mod_{\mathfrak{G}}(\mathcal{D}^{\dagger}_{\mathfrak{X}})$ to be an abelian category so that we can define the associated bounded derived category $D^b_{\mathfrak{G}}(\mathcal{D}^{\dagger}_{\mathfrak{X}})$ of $\mathfrak{G}$-equivariant $\mathcal{D}^{\dagger}_{\mathfrak{X}}$ modules. Another point is the condition of K-coadmissible for compact subgroup K in the definition of an admissible p-adic representation should be translated into some "coherence" condition on the $\mathcal{D}^{\dagger}_{\mathfrak{X}}$-module.

3.2. **Equivariant categories.** Let $\mathcal{X}$ be a (quasi-separated) rigid analytic space whose formal model is a proper smooth formal $\mathcal{V}$ scheme ([BLR 93]). Fix a complete valued field E containing $\mathbb{Q}_p$. Assume that $\mathcal{O}_{\mathcal{X}}$ is an E algebra. The category of sheaves of $\mathcal{O}_{\mathcal{X}}$ modules which are locally convex vector spaces of compact type over E is exact; let $D^b(\mathcal{X})$ be its bounded triangulated category. (Recall that a p-adic locally convex topological space is said to be of compact type if it is a direct limit of Banach spaces with compact transition maps.) Let F be the field of fractions of $\mathcal{V}$. Let G be the F rational points of a reductive group over F. Assume that G acts on $\mathcal{X}$. Then a G equivariant E sheaf is an $\mathcal{O}_{\mathcal{X}}$ module M of locally convex vector spaces of compact type over E together with an $\mathcal{O}_{\mathcal{X}}$ linear isomorphism $b : m^*M \to p^*M$ satisfying the usual cocycle condition given by a commutative diagram as above. The category $D^b(\mathcal{X})$ contains the bounded triangulated category of G equivariant E

sheaves. Let $D^b_G(\mathcal{X})$ be the G-equivariant bounded triangulated category following Bernstein-Lunts (according to Gottker-Schnetmann) and let $\mathcal{F} : D^b_G(\mathcal{X}) \to D^b(\mathcal{X})$ be the forgetful functor such that (1) under $\mathcal{F}$ the image of the heart of $D^b_G(\mathcal{X})$ contains the G equivariant sheaves and (2) $D^b_G(\mathcal{X})$ is equal to $D^b(G\backslash\mathcal{X})$ if the quotient $G\backslash\mathcal{X}$ exists as a rigid space.

3.3. **Overconvergent Solutions.**

3.3.1. As I have taken the liberty to write out the formulas for $SL_2(\mathbb{R})$ I might as well do the same for "overconvergence" for easy reading. Fix a prime p. Let $\mathcal{V}$ be a complete discrete valuation ring of characteristic zero with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{F}$ of characteristic p. Write F for the field of fraction of $\mathcal{V}$. Let $\mathfrak{P}$ be a locally noetherian proper formal $\mathcal{V}$ scheme. Write $P_\mathbb{F}$ for its closed fibre, $\mathcal{P}$ for the rigid analytic space associated to its generic fibre.

Let $sp : \mathcal{P} \to \mathfrak{P}$ be the specialization map according to Berthelot ([Ber 96] (0.2.2)) - locally it is the reduction modulo $\mathfrak{m}$. Let $j : U \to X$ be an open immersion of $\mathbb{F}$-varieties and $X \to P_\mathbb{F}$ be a closed immersion. For example $X = P_\mathbb{F}$. We write this data as a frame $U \xrightarrow{j} X \to \mathfrak{P}$. We write $]U[$ for the inverse image $sp^{-1}(U)$ which we shall call the tube of U. We say a neighbourhood V of $]U[$ in $]X[$ is strict if $]X[= V \cup]X-U[$ is an admissible covering in the rigid topology ([Ber 96] (1.2.1)).

For a sheaf $\mathcal{E}$ on an admissible open subset V of $]X[$ we define the sheaf of germs of sections of $\mathcal{E}$ overconvergent along $X-U$ or simply the overconvergent sheaf

$$j^\dagger\mathcal{E} = j_{V*} \varinjlim j_{VV'*} j_{VV'}{}^*\mathcal{E}$$

where V' runs through all the strict neighbouhoods of $]U[$ in $]X[$, $j_{VV'} : V \cap V' \to V$ and $j_V : V \to]X[$ ([Ber 96] (2.1.1)).

To look at the affine picture let us write t for $(t_1, \cdots, t_n)$. Then the weak completion of the polynomial ring is

$$\mathcal{V}[t]^\dagger = \{\sum a_\alpha t^\alpha \in \mathcal{V}[[t]] : \text{for some } r > 1, |a_\alpha| r^{|\alpha|} \to 0\} ,$$

and the $\mathfrak{m}$-completion of $\mathcal{V}[t]^\dagger$ is

$$\mathcal{V} < t >= \{\sum a_\alpha t^\alpha \in \mathcal{V}[[t]] : \lim a_\alpha = 0\} .$$

Here we use multi-index notation for t^α. Now take $A = \mathcal{V}[t]/(f_1, \cdots, f_m)$, $U = \operatorname{Spec} A$. Let X be the closure of U in the projective space $\mathbb{P}^n_\mathcal{V}$, $\mathfrak{X}$ be the completion along the closed fibre. We have a frame $(U_\mathbb{F} \subset X_\mathbb{F} \subset \mathfrak{X})$. Let us write $B^n(0, r^+)$ for the closed disc of radius r and $\mathcal{U}$ for the rigid analytic space associated to $U \otimes_\mathcal{V} F$. The admissible open sets

$$V_r = B^n(0, r^+) \cap \mathcal{U}$$

form a cofinal family of affinoid strict neighbourhoods of the tube $]U - \mathbb{F}[$ in $]X_\mathbb{F}[$ for $r \geq 1$, $r \to +1$. Put $A_r = \Gamma(V_r, \mathcal{O}_{\mathcal{V}_r})$. Write

$$F < \frac{t}{r} >= \{\sum b_\alpha t^\alpha \in F[[t]] : \lim b_\alpha r^{|\alpha|} \to 0\} .$$

Then

$$A_r = F < \frac{t}{r} > /(f_1, \cdots, f_m)$$

and

$$(A \otimes_{\mathcal{V}} F)^{\dagger} = \lim_{r \to +1} A_r = \mathcal{V}[t]^{\dagger}/(f_1, \cdots, f_m) \otimes_{\mathcal{V}} F \,.$$

Moreover the category of coherent $j^{\dagger}\mathcal{O}_{\mathcal{U}}$-modules is the same as the category of coherent $(A \otimes_{\mathcal{V}} F)^{\dagger}$ - modules . I hope that this helps.

3.3.2. We continue in the notation of section 3.1. Given a formal $\mathcal{V}$ scheme $\mathfrak{X}$ we denote the rigid analytic space associated to the generic fibre of $\mathfrak{X}$ by $\mathfrak{X}_F$ or simply $\mathcal{X}$. To an $\mathcal{O}_{\mathfrak{X}}$ module M we can associate an $\mathcal{O}_{\mathcal{X}}$ module which we denote by M_F. Let $sp : \mathcal{X} \to \mathfrak{X}$ be the specialization map of Berthelot. Let $j^{\dagger}\mathcal{O}_{\mathcal{X}}$ be the sheaf of overconvergent "functions" on $\mathcal{X}$.

Let $\mathcal{M}$ be in the bounded derived category $D^b_{\mathfrak{G}}(\mathcal{D}^{\dagger}_{\mathfrak{X}})$ of $\mathfrak{G}$-equivariant $\mathcal{D}^{\dagger}_{\mathfrak{X}}$ modules. Then applying the solution functor and then taking the associate sheaf over the rigid fibre we get $R\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathfrak{X}}}(\mathcal{M}, sp_* j^{\dagger}\mathcal{O}_{\mathcal{X}})_F$ lying in $D_G(\mathcal{X})$. Applying the global section functor Γ yields representations of G on locally convex topological vector spaces over the p adic fields which we simply denote by $Ext^*(\mathcal{M}, \mathcal{O}_{\mathcal{X}})$. The aim is to find the conditions for "*admissibility*" for $\mathcal{M}$ so that $Ext^*(\mathcal{M}, \mathcal{O}_{\mathcal{X}})$ are admissible representations in the sense of Schneider-Teitelbaum. If this the case and as the Galois group Γ of the algebraic closure of F over F is acting everywhere, we can try to decompose $Ext^*(\mathcal{M}, \mathcal{O}_{\mathcal{X}})$ as an $G \times \Gamma$ space and so obtain a natural setting for pairing p-adic Galois representations with p-adic representations of the p-adic reductive group G.

3.4. **Some definitions.**

3.4.1. It will be convenient to recall the definition of admissible representation here.

Fix a complete p-adic field E. Given a *compact* p-adic Lie group K let $\mathbb{D}(K)$ denote the strong dual of the space of E-valued analytic functions on K. Then $\mathbb{D}(K)$ is an inverse limit of noetherian Banach algebras A_n with *flat* transition homomorphisms. Say a $\mathbb{D}(K)$ module W is K-coadmissible if it is isomorphic to an inverse limit of *finitely generated* A_n modules W_n which are equipped with isomorphisms $A_n \otimes_{A_{n+1}} M_{n+1} \to M_n$.

A representation of a p-adic group G on a locally convex space V over a complete p-adic field E is said to be locally analytic if V is barrelled and for all $v \in V$, the map $G \to V : g \mapsto gv$ is in the space of locally analytic maps from G to V. (For the definition of locally analytic maps from a p-adic manifold to a p-adic locally convex space, see [ST1 02] section 2.) A locally analytic representation of a p-adic group G on a locally convex space V is said to be *admissible* if V is of compact type and the strong dual of V is K-coadmissible for all compact subgroups K of G.

When the compact group K is an union of affinoid open compact subgroups of radii going 1, Emerton ([Eme 04] Prop 5.2.3) identifies the dual of the space of rigid analytic functions on K with a direct limit of p adic completion of finite level divided powers (as in Berthelot [Ber 02]) of the universal enveloping algebra of the Lie algebra of G. This allows to show that $\mathbb{D}(K)$ is the inverse limit of rigid analytic distributions (page 100 end of the proof of Cor 5.3.19) and eventually gives a definition of admissibility equivalent to that of Schneider-Teitelbaum but closer to the theory of arithmetic D modules ([Eme 04] definition6.1.1, Theorem 6.1.20).

3.4.2. I would like to give the definition of the arithmetical differential operators. I agree to it here is not in the logical order but fairy tales never is, besides I feel that it is better to say it here after telling the theme for at this point you will see the familiar features. Before I begin I would also like to recommend to the readers the wonderful papers of Grothendieck ([Gro 66], [Gro 74], Chap IV).

Next we introduce divider powers of level m following Berthelot [Ber 02]). Write $\mathbb{Z}_{(p)}$ for the ring of integers localized at the prime (p). Let A be a $\mathbb{Z}_{(p)}$-algebra. For a given an ideal J of A a divided power structure on J is a set of maps $gamma_n : J \to A\}$ satisfying certain axioms such that $n!\gamma_n(x) = x^n$ for $x \in J$ (see [Gro 74] IV.1 for the full definitions). For example take A to be the quotient ring $\mathbb{Z}/p^m\mathbb{Z}$ for an integer $m > 1$, J to be the ideal generated by p, then $\gamma_n(x) = \frac{x^n}{n!}, x \in J$ defines a divided power structure because $\nu_p(\frac{p^n}{n!}) \geq 1$.

A divided power structure of level m is the data consisting of a $\mathbb{Z}_{(p)}$-algebra A, an ideal J equipped with divided power $gamma_n : J \to A\}$ and an ideal I of A such that $I^{(p^m)} + pI \subset J$. Here we write $I^{(h)}$ for the ideal generated by x^h with $x \in I$. Often say I is the m-PD ideal. For example if for all $x \in I$ we have $x^{p^m} \in pA$ then the ideal $J = pA \cap I$ with the canonical divided power $\gamma_n(x) = \frac{x^n}{n!}$ would give a divided power structure of level m.

Write a positive integer k as $k = r + p^m q$ with $0 \leq r < p^m$, and write $x^{[n]}$ for $\gamma_n(x)$, introduce an operation

$$x^{\{k\}} = x^r (x^{p^m})^{[q]}.$$

The operation $x \mapsto x^{\{k\}}$ enjoys analogous properties as divided powers. Extend the divided power $x^{[n]}$ to $J_1 = J + (p)A$ (and use the same notation) We shall write I_k for the ideal of A generated by $x_1^{\{n_1\}} \cdots x_t^{\{n_t\}}$ for $n_1 + cdots + n_t \geq k$ and write $I^{\{n\}}$ for the ideal of A generated by $x_1^{\{n_1\}} \cdots x_r^{\{n_r\}}\, y_1^{[q_1]} \cdots y_s^{[q_s]}$ with $x_i \in I$, $y_j \in (J + (p)A) \cap I_{k_j}$,and $n_1 + \cdots + n_r + k_1 q_1 + \cdots + k_s q_s \geq n$.

Given a $\mathbb{Z}_{(p)}$-algebra A and any ideal I in A there exists an A-algebra $P_{(m)}(I)$ equipped with a divided power structure $(\bar{I}, \bar{J}, \bar{\gamma})$ of level m satisfying $IP_{(m)} \subset \bar{I}$ which are universal for the homomorphisms from A to A' carrying I into the m-PD ideal of I' of A'. We write

$$P^n_{(m)}(I) = P_{(m)}(I)/\bar{I}^{\{n+1\}}.$$

Let S be a scheme over $\mathbb{Z}/p^{m+1}\mathbb{Z}$ and X be a smooth S scheme. Let $\mathcal{I}$ be the ideal in $\mathcal{O}_{X\times X}$ of the diagonal immersion $X \to X \times_S X$. The construction of $P^n_{(m)}(I)$ leads to the construction of the sheaf $\mathcal{P}^n_{(m)}(\mathcal{I})$ of principal parts of level m of order n. Let $\mathcal{O}_X$ act on it via $f \mapsto f \otimes 1$ (for f in $\mathcal{O}_X$). We define the sheaf of differential operators of level m of order n as

$$\mathcal{D}^{(m)}_{X,n} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{(m)}(\mathcal{I}), \mathcal{O}_X)$$

and the sheaf of differential operators of level m is $\mathcal{D}^{(m)}_X := \cup_{n\geq 0}\mathcal{D}^{(m)}_{X,n}$.

Now let $\mathfrak{X}$ be a smooth formal scheme over a complete discrete valuation ring $\mathcal{V}$ with maximal ideal $\mathfrak{m}$. Put $S_i = Spec(\mathcal{V}/\mathfrak{m}^{i+1})$ and $X_i = S_i \times_{\mathcal{V}} \mathfrak{X}$. Define the sheaf of differential operators of level m on the formal scheme $\mathfrak{X}$ by

$$\widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)} = \varprojlim_i \mathcal{D}_{X_i}^{(m)}$$

and finally the sheaf of arithmetical differential operators on the formal scheme $\mathfrak{X}$ by

$$\mathcal{D}_{\mathfrak{X}}^{\dagger} := \cup_{m \geq 0} \widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)}.$$

On an affine open subset $\mathfrak{U}$ of $\mathfrak{X}$ an element of $\Gamma(\mathfrak{U}, \mathcal{D}_{\mathfrak{X}}^{\dagger})$ can always be written as $\sum a_{\underline{k}} \partial^{\underline{k}} / \underline{k}!$ with $a_{\underline{k}}$ in $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}})$ satisfying $\|a_{\underline{k}}\| < c\eta^{\underline{k}}$ for some real constants c, η independent of $\underline{k}$ and $\eta < 1$.

Just as a reminder let us recall the usual differential operators. For a smooth formal scheme $\mathfrak{X}$ over a complete discrete valuation ring $\mathcal{V}$ let $\mathcal{I}$ be the ideal of the diagonal immersion $\mathfrak{X} \to \mathfrak{X} \times_{\mathcal{V}} \mathfrak{X}$, the sheaf of principal parts of order n is the sheaf $\mathcal{P}^n_{\mathfrak{X}/\mathcal{V}} = \mathcal{O}_{\mathfrak{X} \times \mathfrak{X}} / \mathcal{I}^{n+1}$. Considering $\mathcal{P}^n_{\mathfrak{X}/\mathcal{V}}$ as a $\mathcal{O}_{\mathfrak{X}}$ module via $f \mapsto f \otimes 1$ (for f in $\mathcal{O}_{\mathfrak{X}}$) the sheaf of differential operators of order n is $\mathcal{D}_{\mathfrak{X}/\mathcal{V}, n} := \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{P}^n_{\mathfrak{X}/\mathcal{V}}, \mathcal{O}_{\mathfrak{X}})$. And the sheaf of differential operators is $\mathcal{D}_{\mathfrak{X}/\mathcal{V}} := \cup_{n \geq 0} \mathcal{D}_{\mathfrak{X}/\mathcal{V}, n}$. (This given in EGA IV, 16.8.)

4. Siegel modular varieties

4.1. I would like to begin with a sketch of the moduli problem.

Fix integers $g \geq 2$, $N \geq 3$, a primitive Nth root of unity ζ_N. Let Sch be the category of locally noetherian schemes over $\mathbb{Z}[1/N, \zeta_N]$.

Consider the functor which assigns to a scheme S in Sch the set of isomorphism classes of triples $(A/S, \lambda, \alpha)$ where

1) $A \to S$ is an abelian scheme of relative dimension g.

2) λ is a principal polarization $A/S \to \hat{A}/S$, where $\hat{A}/S$ denotes the dual abelian scheme of A/S.

3) α is a symplectic isomorphism $(A[N], Weil) \to ((\mathbb{Z}/N\mathbb{Z})^g \times \mu_N^g, standard)$ (we call this a level N structure).

This functor is representable by a quasiprojective scheme Y smooth over $\mathbb{Z}[1/N]$ (Mumford [Mum 94], [Cha 85] 1.7).

We choose a smooth toroidal compactification X of Y such that X is a smooth projective scheme over Spec $\mathbb{Z}[1/N, \zeta_N]$ together with an action of $Sp_{2g}(\mathbb{Z}[1/N, \zeta_N)$ ([Ash 75], [FaC 90] V section5).

Fix a prime p not dividing N. Let $\mathcal{V}$ be the completion of $\mathbb{Z}[1/N, \zeta_N]$ at a prime $\mathfrak{p}$ above p. Let $\mathbb{F}$ (resp. F) be the residue field (resp. field of fractions) of $\mathcal{V}$. Let $\mathfrak{X}$ be the formal completion of X along the closed fibre over $\mathfrak{p}$.

4.2. Let G be the semisimple split symplectic group scheme of rank g over $\mathbb{Z}$. We can take

$$G(\mathbb{Z}) = \{\gamma \in GL_{2g}(\mathbb{Z}) : {}^t\gamma J \gamma = \}$$

with $J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$. Write Γ for the congruence subgroup of level N, i.e. $\Gamma = \mathrm{Ker}(G(\mathbb{Z}) \to G(\mathbb{Z}/N\mathbb{Z}))$. Let K be the maximal compact subgroup of $G(\mathbb{R})$. Then the complex points of Y is

$$Y(\mathbb{C}) = K\backslash G(\mathbb{R})/\Gamma\ .$$

In fact let $\mathfrak{G}$ be the formal completion of G at $\mathfrak{p}$. Then we have an action of $\mathfrak{G}$ on $\mathfrak{X}$. Suppose we have a system of differential equations or better an $\mathcal{M}$ in the bounded derived category $D^b_{\mathfrak{G}}(\mathcal{D}^\dagger_{\mathfrak{X}})$ of $\mathfrak{G}$-equivariant $\mathcal{D}^\dagger_{\mathfrak{X}}$ modules, then on the solution spaces $Ext^*(\mathcal{M}, \mathcal{O}_{\mathfrak{X}})$ the p-adic symplectic group $G(F)$ acts.

4.3. Write $X_{\mathbb{F}}$ for the closed fibre of $\mathfrak{X}$. Let U be the ordinary locus in $Y_{\mathbb{F}}$. We have a frame $U \xrightarrow{j} X_{\mathbb{F}} \to \mathfrak{X}$. Let $\mathcal{M}$ be in the bounded derived category $D^b_{\mathfrak{G}}(\mathcal{D}^\dagger_{\mathfrak{X}})$ of $\mathfrak{G}$-equivariant $\mathcal{D}^\dagger_{\mathfrak{X}}$ modules. The problem is to study the admissibility of the $Sp_{2g}(\mathbb{Q}_p)$ representations on the $Ext^*(\mathcal{M}, \mathcal{O}_{\mathcal{X}})$ obtained from the overconvergent solutions $R\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{X}}}(\mathcal{M}, sp_* j^\dagger \mathcal{O}_{\mathcal{X}})_F$.

As the Galois group $Gal(\bar{F}/F)$ also acts on $Ext^*(\mathcal{M}, \mathcal{O}_{\mathcal{X}})$ via its action on $\mathcal{X}$ as a moduli space, it would be interesting to study $Ext^*(\mathcal{M}, \mathcal{O}_{\mathcal{X}})$ as a $Gal(\bar{F}/F) \times Sp_{2g}(\mathbb{Q}_p)$ module; in particular the relation with p adic Hodge theory via the Beerger-Kisin functor between the (ϕ, N)-modules over K_0 and the (ϕ, N_∇)-modules over $\mathcal{O}$ as in the original work of Berger-Breuil.

4.4. Write $\pi : A \to Y$ for the universal abelian scheme. Put $\omega = \omega_{Y/A} = \pi_*(\wedge^g \Omega^1_{A/Y})$. The elements of $H^0(Y, \omega^{\otimes k}_{A/Y})$ are Siegel modular forms of weight k and level N.

The universal abelian scheme $A \to Y$ extends to a semiabelian scheme $\bar{A} \to X$. Write $\bar{\omega}$ for the extension of ω to X ([Mum 77]). According to Koecher's principle we have

$$H^0(Y, \omega^{\otimes k}) = H^0(X, \bar{\omega}^{\otimes k}).$$

Let $\mathcal{X}$ be the rigid analytic space associated to $\mathfrak{X}$. We shall use the same notation for the sheaf on $\mathcal{X}$ associated to the sheaf $\bar{\omega}$ on X. We can twist the $(\mathbb{G}_m/K)$-torsor $\bar{\omega}$ on $\mathcal{X}$ on $\mathcal{X}$ by pushing forward along a a character $\kappa : \mathbb{G}_m/K \to \mathbb{G}_m/K$ and we denote the resulting sheaf by $\bar{\omega}^\kappa$.

We now apply Berthelot's construction to obtain $j^\dagger \bar{\omega}^\kappa$. The rigid analytic space on $Y \otimes K$ is denoted by $\mathcal{Y}$. We call an element of $M_\kappa(N)^\dagger = H^0(\mathcal{Y}, j^\dagger \bar{\omega}^\kappa)$ an overconvergent Siegel modular form of weight κ and level N.

It would be nice to know that $j^\dagger \bar{\omega}^\kappa$ is independent of the choice of the lattice in the polyhedral decomposition used for the construction of the integral model of the toroidal compactification. As it is we shall simply take the direct limit of $j^\dagger \bar{\omega}^\kappa$ over compactifications build from a tower of lattices to allow the actions of Hecke operators as constructed in [FaC 90] (see also the theis of KeiWen Lan).

We have a natural $\mathcal{D}^\dagger_{\mathfrak{X}}$-module structure on $sp_* j^\dagger \bar{\omega}^\kappa$. Are the representations of the p-adic symplectic group on the solution spaces $Ext^*(sp_* j^\dagger \bar{\omega}^\kappa, \mathcal{O}_{\mathcal{X}})$ admissible?

We can also ask if we take as M the sheaf of overconvergent p-adic Siegel modular forms or $j^\dagger$ of automorphic bundles ([Har 856], [Mil 88]) what do we get?

4.5. I agree that it would be easier to begin with the rank one case namely that of SL_2. It would be fantastic to establish Breuil's conjecture on p adic Langlands correspondence for GL_2 via arithmetic $\mathcal{D}$-modules.

On the other hand it has also been suggested that one could look at the universal deformation space of one dimensional formal group of height h together with its period map into the $h-1$ dimensional projective space. But such formal schemes are neither of finite type nor proper over the base scheme and so the standard assumptions in Berthelot are not satisfied. May be there is some way to overcome this.

4.6. Though it is said in a different setting I would still like to recall Varadarajan's comment in the real case: "...resort to the hathayoga of special functions to do everything...will be an exercise in futility for it will tell us almost nothing of what is likely to happen in the general case." ([VSV 89] p.226.) In another context in which Katz studied p-adic L functions by applying p-adic differential operators to Eisenstein series, he wrote : "we were able to prove theorems by dipping into classical material... We were for a long time blinded by these riches to the simple cohomological mechanism which in some sense underlies them." ([Kat 78] p.205.)

References

[Art 89] J. Arthur, *Unipotent automorphic representation* Asterisque **171** (1989) 13-71.

[Art 02] J. Arthur, *A note on the automorphic Langlands group* Canad. Math. Bull. **45** (2002) 466-482.

[Art 03] J. Arthur, *The principle of functoriality*, Bull. AMS **40** (2003) 39-53.

[Ash 75] A. Ash, D. Mumford, M. Rapoport, Y. Tai, *Smooth compactification oflocally symmetric varieties*, (1975) Math.Sci.Press, Boston.

[AtS 77] M. Atiyah, W. Schmid , *A geometric construction of discrete series*, Inv. Math. **42** (1977) 1-62.

[AuC 04] A-M. Aubert, C. Cuningham, *Depth zero character sheaves*, preprint math.jussieu.fr (2004).

[Bar 47] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Ann. of Math. **48** (1947) 568-640.

[BeB 81] A. Beilinson, J. Bernstein, *Localisation de $\mathfrak{g}$-modules*, C.R. Acad. Sci. Paris **292** (1981) 15-18.

[BBr 06] L. Berger, C. Breuil, *Sur les representations potentiellement cristallines de $GL_2(\mathbb{Q}_p)$*, preprint, umpa.ens-lyon, 2006.

[Beg 04] L. Berger, *Equations differentielles p-adiques et (ϕ, N) modules filtres*, preprint umpa.ens-lyon, (2004).

[Ber 96] P. Berthelot, *Cohomologie rigide et cohomologie rigide à supports propre. Première partie*, Prépublication IRM 96-08 , 91 pages, 1996.

[Ber 961] P. Berthelot, *D-modules arithmetiques I*, Ann. scient. Eco. Norm. Sup. **29** (1996) 185-272 .

[Ber 20] P. Berthelot, *D-modules arithmetiques II*, Mem. Soc. Math. France **81** (2000).

[Ber 02] P. Berthelot, *Introduction a la theorie arithmetique des D-modules*, Asterisques **279** (2002) 1-80.

[BeV 88] J. Bernstein, V. Lunts, *On nonholonomic irreducible D modules*, Inv. Math. **94** (1988) 223-243.

[BeV 94] J. Bernstein, V. Lunts, *Equivariant sheaves and functors*, Lecture Notes Math.**1578** (1994) Springer.

[Bjo 93] J-E. Bjork, *Analytic D modules and applications*, Kluwer Academic Press (1993).

[Bor 87] A. Borel, *Algebraic D modules* Academic Press (1987).

[BLR 93] S. Bosch, W. Lütkebohmert , M. Raynaud, *Formal and rigid geometry*, I Math. Ann. **295** (1993) 291-317, II Math. Ann. **296** (1993) 403-429, III Math.Ann. **302** (1995) 1-29.

[Bre 03] C. Breuil, *Sur quelques representations modulaires et p-adiques de* $GL_2(\mathbb{Q}_p)$, I Compo. Math. **138** (2003) 165-188, II J. Inst. Math. Jussieu, **2** (2003) 23 -58.

[BrS 03] C. Breuil, P. Schneider, *First steps towards p-adic Langlands functoriality*, Preprint, Muenster.

[Car 00] H. Carayol, *Preuve de la conjecture de Langlands locale pour* GL_n, Sem. Bourbaki, Asterisque **266** (2000) 191- 243.

[Cha 85] C-L. Chai, *Compactification of Siegel moduli scheme*, Cambridge Univ. Press, Cambridge, 1985.

[Dwo 73] B. Dwork, *On p-adic differential equations*, I Soc. Math¿ Fr. Mem. **39-40** (1974) 27-37; II Ann. of Math. **98** (1973) 366-376; III Inv. Math. **20** (1973) 35-45; IV Ann. Eco. Norm. Sup. **6** (1973) 295-315.

[Eme 04] M. Emerton, *Locally analytic vectors in representations of locally p-adic analytic groups*, preprint math.northwestern.edu (2004).

[FaC 90] G. Faltings, C.-L. Chai *Degeneration of abelian varieties*, (1990) Springer.

[Fon 90] J-M. Fontaine, *Representations p-adiques des corps locaux I*, Birkauser Prog. Math. **87** (1990) 249-309.

[Fre 04] E. Frenkel, *Recent advances in the Langlands program*, Bull. AMS **41** (2004) 151- 184.

[GKM 98] M. Goresky,R. Kottwitz, R. MacPherson, *Equivariant cohomology, Koszul duality and the localization theorem*, Inven. Math. **131** (1998) 25-83.

[Got 04] J. Gottker-Schnetmann, *Aquivqriante derivierte Kategorien rigider Raume*, Schriftenreihe Math. Inst. Muenster 3 ser. **31** (2004) Univ. Muenster.

[Gro 01] M. Gros, *Caracters des groupes redutifs fini et D-modules*, REnd. Sem. Mat. Univ. Padova **106** (2001) 1-19.

[Gro 66] A. Grothendieck, *On the de Rham cohomology of algebraic varieties*, Pub Math IHES**29** (1966) -.

[Gro 74] A. Grothendieck,, *Groupes de Barsotti-Tate et cristaux de Dieudonne*, Press de l'universite de Montreal (1974).

[GrH 94] B.H. GRoss, M. J. Hopkins, *Equivariant vector bundles on the Lubin-Tate moduli space*, in Topology and Representation Theory, AMS Contemp. Math. **158** (1994) 23-88.

[Har 856] M. Harris, *Arithmetic vector bundles and automorphic forms on Shimura varieties*, I Inv. Math. **82** (1985) 151-189, II Comp. Math. **60**(1986) 323-378.

[HMS 87] H. Hecht, D. Milicic, W. Schmidt, J. Wolf, *Localization and standard modules for real semisimple Lie groups I*, Inv. Math. **90** (1987) 297-332.

[HoK 84] R. Hotta, M. Kashiwara, *Invariant holonomic system on a semisimple algebra*, Inv. Math. **75** (1984) 327-358.

[HoK 87] R. Hotta, M. Kashiwara, *Quotients of the Harish-Chandra systems by primitive ideals*, in Giornate di geometria, Prog. Math. **60** (1985) 185-205.

[Hot 87] R. Hotta, *Holonomic D modules in representation theory*, Proc. Sympos. Pure Math. AMS **47-1** (1987) 87-102.

[Kas 89] M. Kashiwara, *Representation theory and D-modules on flag varieties*, Asterisque **173-4** (1989) 55-109.

[KaS 00] M. Kashiwara, W. Schmidt *Quasi-equivariant D-modules, equivariant derived category and represntations of reductive Lie groups*, in "Lie theory and geometry" Prog Math 123 (1994) 457-488, Birkhauser.

[KaT 84] M. Kashiwara, T. Tanisaki, *The characteristic cycles of holonomic systems on flag varieties*, Inv. Math. **77** (1984) 185-198.

[Kat 76] N. Katz, *p-adic interpolation of real analytic Eisenstein series*, Ann. of Math. **104** (1976) 459-571.

[Kat 78] N. Katz, *p-adic L functions for CM fields*, Inven. Math. **49** (1978) 199-297.

[KaY 07] M. Kaneda, J. Ye, *Equivariant localization of D modules on the flag variety of the symplectic group of degree 4*, J. Alg. **309** (2007) 236-281.

[Ked 05] K. S. Kedlaya, *Local monodromy of p-adic differential equations: an overview*, arXiv: :math.NT/050136 v2 (2005).

[KoS 99] R. Kottwitz, D. Shelstad, *Foundations of Twisted Endoscopy*, Asterisque **255** (1999).

[LaL 79] J.-P. Labesse, R.P. Langlands, *L-indistinguishability for* SL_2, Canad. J. Math. **31** (1979) 726-785.

[LRS 93] G. Laumon, M. Rapoport, U.Stuhler, *D-elliptic sheaves and teh Langlands correspondence*, Inven. Math. **113** (1993) 217-338.

[Lau 95] G. Laumon, *Faisceaux automorphes pour* GL_n, arXiv: : alg-geom/9511004v1 (1995).

[Lau 02] G. Laumon, *La correspondence de Langlands sur les corps de fonctions*, Sem. Bourbaki Asterisque **276** (2002) 207-265.

[Lut 90] W. Lütkebohmert *Formal-algebraic and rigid analytic geometry*, Math. Ann. **286** (1990) 341-371.

[Mil 88] J.S. Milne, *Automorphic vector bundles on connected Shimura varieties*, Inven. Math. **92** (1988) 91-128.

[MiP 98] D. Milicic, P. Pandzic, *Equivariant derived categories, Zuckerman functors and localization*, in "Geometry and representation theory of real and p-adic groups" Prog Math 158 (1998) 209-242, Birkhauser.

[Mum 77] D. Mumford, *Hirzebruch's proportionality theorem in the noncompact case*,Inven. Math. **42** (1977) 239-272.

[Mum 94] D. Mumford, J. Fogarty, F. Kirwan, *Geometric invariant theory.* Third edition. (1994) Springer-Verlag, Berlin.

[Pan 04] A. Panchishkin, M. Courtieu, *Non-archimedean L functions and arithmetic Siegel modular forms*, Springer Lecture Notes in Math. **1471** second edition (2004).

[Put 86] M. van der Put, *The cohomology of Monsky and Washinitzer*, Mem. Soc. Math. France **23** (1986) 33-59.

[Sch 71] W. Schmid, *On a conjecture of Langlands*, Ann. of Math. **93** (1971) 1-42.

[Sch 76] W. Schmid, L^2 *cohomology and the discrete series*, Ann. of Math. **103** (1976) 375-394.

[ScW 90] W. Schmid, J. Wolf, *Geometric quantization and derived functor modules for semisimple Lie groups*, J. Func. Anal. **90** (1990) 48-112.

[Shi 81] G. Shimura, *Arithmetic of differential operators on bounded symmmetric domains*, Duke M. J. **48** (1981) 813-843.

[ST1 02] P. Schneider, J. Teitelbaum, *Locally analytic distributions and p-adic representation theory with applications to* GL_2, JAMS **15** (2002) 443- 468.

[ST2 02] P. Schneider, J. Teitelbaum, *Banach space representations and Iwasawa theory*, Israel J. Math. **127** (2002) 359-380.

[VSV 77] V.S. Varadaranjan, *Harmonic analysis on real reductive groups*, Lecture Notes in Math. **576** (1977) Springer.

[VSV 89] V.S. Varadaranjan, *Harmonic analysis on semisimple Lie groups*, (1989), Cambridge University Press.

[VSV 96] V.S. Varadaranjan, *Linear meromorphic differential equations*,Bull. AMS **33** (1996) 1-42.

Castle Peak Road, Tuen Mun, Hong Kong
E-mail address: rexinterra@gmail.com